

Bounds on Distance to Variety in Terms of Coefficients of Bivariate Polynomials

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Let $f \in \mathbb{C}[x]$ be a univariate polynomial of degree d with roots $\alpha_1, \dots, \alpha_d$. For a point $z \in \mathbb{C}$, let $\text{sep}(f, z) := \min_i |z - \alpha_i|$. Then for all points $z \in \mathbb{C}$ we know that the logarithmic derivative at z is

$$\frac{f'(z)}{f(z)} = \sum_{i=1}^d \frac{1}{|z - \alpha_i|}. \quad (1)$$

and more generally for any $k \geq 1$ we have

$$\frac{f^{(k)}(z)}{f(z)} = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq d} \frac{1}{(z - \alpha_{i_1})(z - \alpha_{i_2}) \dots (z - \alpha_{i_k})}. \quad (2)$$

Taking absolute value on both sides, applying triangular inequality on the RHS, and observing that the number of terms on the RHS is $\binom{d}{k}$ and each is smaller than $1/\text{sep}(f, z)$ we get the following bound: for $k \geq 1$

$$\left| \frac{f^{(k)}(z)}{f(z)} \right|^{1/k} \leq \frac{d}{\text{sep}(f, z)}. \quad (3)$$

Another way to interpret this bound is to state it as follows:

$$\text{sep}(f, z) \leq d \min_{1 \leq k \leq d} \left| \frac{f(z)}{f^{(k)}(z)} \right|. \quad (4)$$

Similar bounds are also derived in [Hen74, p. 452]. In this short note, we will generalize this result to bivariate polynomial $f(x, y) \in \mathbb{C}[x, y]$. The analogue result will have the following form: the left hand side will be the distance of a point \mathbf{p} from the variety of f , and the RHS will consist of the total degree of f and a quantity dependent on the absolute values of f and its partial derivatives evaluated at \mathbf{p} . We first establish some notation. For $k \geq 0$, define

$$f_{i,k}(\mathbf{p}) := \frac{\partial^k f(\mathbf{p})}{\partial^i x \partial^{k-i} y}. \quad (5)$$

Let D be the total degree of f , $V(f) \subseteq \mathbb{C}^2$ be the variety of f , and

$$\text{sep}(\mathbf{p}, V(f)) := \inf_{\mathbf{x} \in V(f)} \|\mathbf{p} - \mathbf{x}\| \quad (6)$$

be the distance function to $V(f)$.

The idea for deriving the bound is as follows. Consider a point $\mathbf{p} = (\mathbf{p}_x, \mathbf{p}_y) \in \mathbb{C}^2 \setminus V(f)$. In order to derive an upper bound on $\text{sep}(\mathbf{p}, V(f))$, we will consider all the lines through \mathbf{p} . These lines intersect the curve $f(x, y) = 0$ at finitely many points that can be obtained as roots of a univariate polynomial. For instance, consider the intersection of the line $x = \mathbf{p}_x$ with the curve $f = 0$. Apply the upper bound in (4) to the resulting univariate polynomial we obtain that

$$\text{sep}(\mathbf{p}, V(f)) \leq D \min_{1 \leq k \leq D} \left| \frac{f(\mathbf{p})}{f_{0,k}(\mathbf{p})} \right|^{1/k}.$$

Similarly, considering the intersection of the line $y = \mathbf{p}_y$ with the curve $f = 0$ we also get that

$$\text{sep}(\mathbf{p}, V(f)) \leq \min_{1 \leq k \leq D} \left| k! \binom{D}{k} \frac{f(\mathbf{p})}{f_{k,0}(\mathbf{p})} \right|^{1/k}.$$

How do we get the terms corresponding to the mixed partial derivatives? We consider all the lines with slope $\tan \theta$, as θ varies from 0 to 2π , and take the minimum of the absolute value of the corresponding roots over all θ . Since this function is periodic in θ , it makes sense to use some tools from Fourier analysis. The remaining section develops this idea into full detail.

Considering f as a polynomial in x with coefficients in $\mathbb{C}[y]$, from the local parameterization of algebraic curves [Wal78], we know that in a certain neighborhood of a point $(x, y) \in \mathbb{C}^2 \setminus V(f)$ we can express

$$f(x, y) := K \prod_{i=1}^{d(y)} (x - \alpha_i(y)), \quad (7)$$

where α_i 's are holomorphic functions of y , the degree $d(y) \leq \deg(f, x)$ depends on the y -coordinate, and $K \in \mathbb{C}$ is some constant. Differentiating both sides with respect to x and factoring $f(x, y)$ from the RHS we obtain that

$$f_{1,0}(x, y) = f(x, y) \sum_{i=1}^{d(y)} \frac{1}{x - \alpha_i(y)}, \quad (8)$$

and in general

$$f_{k,0}(x, y) = f(x, y) \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq d(y)} \frac{1}{(x - \alpha_{i_1}(y)) \dots (x - \alpha_{i_k}(y))}.$$

Following the argument used to derive (3) in the univariate setting, we obtain that for any point $\mathbf{p} \in \mathbb{C}^2$

$$\left| \frac{f_{k,0}(\mathbf{p})}{f(\mathbf{p})} \right|^{1/k} \leq \frac{D}{\text{sep}(\mathbf{p}, V(f))}, \quad (9)$$

Note that if there is an asymptote at y then $d(y) < \deg(f, x)$; also, if $d(y) = 0$ then the bound above on the partial derivatives trivially holds since all the partial derivatives vanish.

We want to derive a similar bound for the mixed partial derivatives $f_{i,k-i}(\mathbf{p})$. To obtain this, we change the coordinate system and then consider the intersection with either the horizontal or vertical axis. Consider the following change of coordinates:

$$\begin{bmatrix} x \\ y \end{bmatrix} := \frac{1}{\sqrt{2}} \begin{bmatrix} e^{j\theta} & e^{-j\psi} \\ e^{-j\psi} & -e^{-j\theta} \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = U \cdot \begin{bmatrix} X \\ Y \end{bmatrix}$$

where $j = \sqrt{-1}$ and θ, ψ are any angles; we will later set $\psi = 0$. Note that the matrix U is unitary since

$$UU^\dagger = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{j\theta} & e^{-j\psi} \\ e^{j\psi} & -e^{-j\theta} \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} e^{-j\theta} & e^{j\psi} \\ e^{-j\psi} & -e^{j\theta} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Define

$$F(X, Y) := f(U(X, Y)) = f\left(\frac{e^{j\theta}X + e^{-j\psi}Y}{\sqrt{2}}, \frac{e^{j\psi}X - e^{-j\theta}Y}{\sqrt{2}}\right).$$

By repeated applications of the chain rule of partial differentiation we know that

$$F_{k,0}(X, Y) = \sum_{i=0}^k \binom{k}{i} (f_{i,k-i} \circ U(X, Y)) \left(\frac{\partial x}{\partial X}\right)^i \left(\frac{\partial y}{\partial X}\right)^{k-i}.$$

Observe that

$$\partial x / \partial X = e^{j\theta} / \sqrt{2}, \quad \partial y / \partial X = e^{j\psi} / \sqrt{2}.$$

Thus,

$$F_{k,0}(X, Y) = \sum_{i=0}^k \binom{k}{i} (f_{i,k-i} \circ U(X, Y)) e^{ji\theta} 2^{j(k-i)\psi} 2^{-k}. \quad (10)$$

Since total degree of F is the same as the total degree of f , it follows from (9) that for a point $\mathbf{p} \in \mathbb{C}^2$

$$\left| \frac{F_{k,0}(U^{-1}(\mathbf{p}))}{F(U^{-1}(\mathbf{p}))} \right|^{1/k} \leq \frac{D}{\text{sep}(U^{-1}(\mathbf{p}), V(F))} = \frac{D}{\text{sep}(U^{-1}(\mathbf{p}), U^{-1}(V(f)))} = \frac{D}{\text{sep}(\mathbf{p}, V(f))}, \quad (11)$$

where the last step follows from the fact that U is a unitary transformation.

Moreover, as $F(U^{-1}(\mathbf{p})) = f(\mathbf{p})$, from (10) and (11) we obtain that for all choices of $\theta \in [-\pi, \pi]$

$$\left| \sum_{i=0}^k \binom{k}{i} \frac{f_{i,k-i}(\mathbf{p})}{f(\mathbf{p})} e^{ji\theta} e^{j(k-i)\psi} 2^{-k} \right|^{1/k} \leq \frac{D}{\text{sep}(\mathbf{p}, V(f))}. \quad (12)$$

Let $P(\theta)$ be the function inside the absolute value on the LHS above. Since it is a Fourier series in θ , from Parseval's theorem we know that

$$\sum_{i=0}^k \left(\binom{k}{i} \left| \frac{f_{i,k-i}(\mathbf{p})}{f(\mathbf{p})} \right| e^{j(k-i)\psi} 2^{-k} \right)^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |P(\theta)|^2 d\theta.$$

Substituting the upper bound (12) on $|P(\theta)|$ in the integral on the RHS above we further obtain that

$$\sum_{i=0}^k \left(\binom{k}{i} \left| \frac{f_{i,k-i}(\mathbf{p})}{f(\mathbf{p})} \right| e^{j(k-i)\psi} 2^{-k} \right)^2 \leq \left(\frac{D}{\text{sep}(\mathbf{p}, V(f))} \right)^{2k}. \quad (13)$$

Choosing $\psi = 0$, we obtain

$$\sum_{i=0}^k \left(\binom{k}{i} \left| \frac{f_{i,k-i}(\mathbf{p})}{f(\mathbf{p})} \right| e^{j(k-i)\psi} 2^{-k} \right)^2 > \max_{j=0,\dots,k} \left(\left| \frac{f_{j,k-j}(\mathbf{p})}{f(\mathbf{p})} \right| 2^{-k} \right)^2$$

Combining this with (13) we obtain the following:

THEOREM 1. *For a point $\mathbf{p} \in \mathbb{C}^2$ that is not a zero of f*

$$\max_{i=0,\dots,k} \left| \frac{f_{i,k-i}(\mathbf{p})}{f(\mathbf{p})} \right|^{1/k} < \frac{2D}{\text{sep}(\mathbf{p}, V(f))}. \quad (14)$$

The bound above can also be interpreted as an upper bound on the separation of a point \mathbf{p} from the variety $V(f)$ in terms of the coefficients of the polynomial. Can a converse bound be given, i.e., a *lower bound* on the separation in terms of the coefficients. We next derive such a bound.

Suppose $f(0) \neq 0$, then we want to derive a lower bound on $\text{sep}(0, V(f))$ in terms of the coefficients. Clearly, any $\mathbf{x} = (x, y)$ for which

$$|a_{0,0}| > \sum_{k \geq 1} \sum_{i=0}^k |a_{i,k-i}| |x|^i |y|^{k-i} \quad (15)$$

cannot be on the variety of f . Define

$$\gamma := \max_{1 \leq k \leq D} \max_{0 \leq i \leq k} \left(\frac{k!}{\binom{k}{i}} \left| \frac{a_{i,k-i}}{a_0} \right| \right)^{1/k},$$

where D is the total degree. Then it follows that (15) is equivalent to

$$1 > \sum_{k \geq 1} \frac{\gamma}{k!} \sum_{i=0}^k \binom{k}{i} |x|^i |y|^{k-i} = \sum_{k \geq 1} \frac{\gamma}{k!} (|x| + |y|)^k > \exp(\gamma \|\mathbf{x}\|_1) - 1.$$

Therefore, if \mathbf{x} is such that $\|\mathbf{x}\|_1 \gamma < \ln 2$ then $|f(\mathbf{x})| > 0$. In general, for any point $\mathbf{p} \in \mathbb{C}^2$ we can apply the argument above to the shifted polynomial to obtain the following: if

$$\gamma_f(\mathbf{p}) := \max_{1 \leq k \leq D} \max_{0 \leq i \leq k} \left| \frac{f_{i,k-i}(\mathbf{p})}{f(\mathbf{p})} \right|^{1/k}, \quad (16)$$

then

$$\text{sep}(\mathbf{p}, V(f)) \geq \frac{\ln 2}{\sqrt{2}\gamma(\mathbf{p})} \geq \frac{1}{3\gamma(\mathbf{p})}. \quad (17)$$

Besides their intrinsic interest, such bounds are useful in analyzing the complexity of certain algorithms. For instance, the bound given in (3) has been useful in bounding the running time of certain root isolation algorithms using the continuous amortization framework [Bur16, SB15]. We expect the generalization given above to be useful in deriving similar bounds on the running time of generalizations of corresponding algorithms that generally use subdivision (e.g., [PV04]).

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